Valuation of Interest-Bearing and Non-Interest-Bearing Liabilities
Part I - Company Revenue Growth Rate is Constant

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In this white paper we will build a model to value interest-bearing liabilities (IBL) and non-interest-bearing liabilities (NIBL). IBL consists primarily of interest-bearing debt and capitalized lease obligations. NIBL consists primarily of accounts payable, accrued expenses, deferred tax liabilities and other liabilities.

To value IBL we will construct a model to value interest-bearing liabilities where term is finite or infinite and the principal balance can increase or decrease at a constant rate over time. We will then use the IBL valuation model to value NIBL where term is infinite. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the market value of a company's liabilities. These liabilities can be either interest-bearing (IBL) or non-interest-bearing (NIBL) and both are unsecured for bankruptcy purposes and therefore are subject to the same risk-adjusted discount rate. Our go-forward model assumptions are...

Description	Value
Annualized revenue (in dollars)	1,000,000
Annualized revenue growth rate (percent)	3.00
Ratio of assets to annualized revenue	1.20
Ratio of liabilities to assets	0.45
Annualized interest accrual rate (percent)	4.00
Annualized coupon payment rate (percent)	2.00
Annualized risk-adjusted discount rate (percent)	6.00

Question 1: What is the balance of total liabilities at time zero?

Question 2: What is the market value of IBL at time zero given that term is 15 years?

Question 3: What is the market value of IBL at time zero given that term is perpetual?

Question 4: What is the market value of NIBL at time zero given that term is perpetual?

IBL Equations

We will define the variable μ to be the continuous-time revenue growth rate. The equation for the revenue growth rate that is constant over time is...

$$\mu = \ln \left(1 + \text{annualized revenue growth rate} \right) \tag{1}$$

We will define the variable R_t to be annualized revenue at time t. Using Equation (1) above the equation for annualized revenue at time t as a function of annualized revenue at time zero is...

$$R_t = R_0 \operatorname{Exp} \left\{ \mu \, t \right\} \tag{2}$$

We will define the variable A_t to be total assets at time t and the variable θ to be the ratio of total assets to annualized revenue. Using Equation (2) above the equation for total assets at time t as a function of annualized revenue at time zero is...

$$A_t = \theta R_t = \theta R_0 \operatorname{Exp}\left\{\mu t\right\} \tag{3}$$

We will define the variable P_t to be liability principal balance (excludes accrued interest payable) at time t and the variable ϕ to be the ratio of total liabilities to total assets. Using Equation (3) above the equation for liability principal balance is...

$$P_t = \phi A_t = \phi \theta R_0 \operatorname{Exp}\left\{\mu t\right\} \text{ ...such that... } \delta P_t = \mu \phi \theta R_0 \operatorname{Exp}\left\{\mu t\right\} \delta t \tag{4}$$

We will define the variable ω to be the continuous-time interest accrual rate. The equation for the interest accrual rate is...

$$\omega = \ln\left(1 + \text{annualized interest accrual rate}\right) \tag{5}$$

We will define the variable α to be the continuous-time coupon payment rate. The equation for the coupon payment rate is...

$$\alpha = \ln\left(1 + \text{annualized coupon payment rate}\right) \tag{6}$$

We will define the variable I_t to be accrued interest payable at time t. Using Equations (4), (5) and (6) above the equation for accrued interest payable is...

$$I_t = \int_0^t (\omega - \alpha)\, \phi\, \theta R_0 \operatorname{Exp}\left\{\mu s\right\} \operatorname{Exp}\left\{\omega\,(t-s)\right\} \delta s \text{ ...given that... } I_0 = 0 \tag{7}$$

Note that in Equation (7) above the first half of that equation accrues interest on liability principal balance at time s and the second half of that equation accounts for the compounding of interest over the time interval $[s,t]$. Using Appendix Equation (38) below the solution to Equation (7) above is...

$$I_t = (\omega - \alpha)\left(\frac{\phi\,\theta R_0}{\mu - \omega}\right)\left(\operatorname{Exp}\left\{\mu t\right\} - \operatorname{Exp}\left\{\omega t\right\}\right) \tag{8}$$

We will define the variable IBL_t to be total interest-bearing liabilities (principal plus accrued interest) at time t. Using Equations (4) and (8) above the equation for interest-bearing liabilities at time t is...

$$
\begin{aligned}
IBL_t &= P_t + I_t \\
&= \phi\,\theta R_0 \operatorname{Exp}\left\{\mu t\right\} + (\omega - \alpha)\left(\frac{\phi\,\theta R_0}{\mu - \omega}\right)\left(\operatorname{Exp}\left\{\mu t\right\} - \operatorname{Exp}\left\{\omega t\right\}\right) \\
&= \phi\,\theta R_0\left[\operatorname{Exp}\left\{\mu t\right\} + \frac{\omega - \alpha}{\mu - \omega}\left(\operatorname{Exp}\left\{\mu t\right\} - \operatorname{Exp}\left\{\omega t\right\}\right)\right]
\end{aligned} \tag{9}
$$

IBL Valuation Equations

Over the time interval $[0, T]$ the lender receives coupon payments from the borrower and either disburses (liability principal increases) or receives (liability principal decreases) the change in principal to/from the borrower. Using Equations (4), and (6) above the equations for the two components of lender cash flow over the infinitesimally small time interval $[t, t + \delta t]$ are...

$$\text{Change in principal}: \mu\, P_t\, \delta t = \mu\, \phi\, \theta R_0 \operatorname{Exp}\left\{\mu t\right\} \delta t \quad \Big| \quad \text{Coupon payments}: \alpha\, P_t\, \delta t = \alpha\, \phi\, \theta R_0 \operatorname{Exp}\left\{\mu t\right\} \delta t \tag{10}$$

We will define the variable C_t to be lender net cash flow over the time interval $[t, t + \delta t]$. The equation for lender net cash flow prior to maturity is...

$$C_t = \text{Period coupon payments} - \text{Period increase/(decrease) in principal} \text{ ...where... } 0 < t < T \tag{11}$$

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Using Equation (10) above we can rewrite Equation (11) above as...

$$C_t = \alpha \, \phi \, \theta R_0 \operatorname{Exp}\left\{\mu \, t\right\} \delta t - \mu \, \phi \, \theta R_0 \operatorname{Exp}\left\{\mu \, t\right\} \delta t = (\alpha - \mu) \, \phi \, \theta R_0 \operatorname{Exp}\left\{\mu \, t\right\} \delta t \tag{12}$$

We will define the variable C_T to be lender cash flow at maturity (i.e. at time T). The equation for lender cash flow at maturity is...

$$C_T = \text{Principal balance at time } T + \text{Accrued interest payable at time } T \tag{13}$$

Using Equation (9) above we can rewrite Equation (13) above as...

$$C_T = \phi \, \theta R_0 \left[\operatorname{Exp}\left\{\mu \, T\right\} + \frac{\omega - \alpha}{\mu - \omega} \left(\operatorname{Exp}\left\{\mu \, T\right\} - \operatorname{Exp}\left\{\omega \, T\right\} \right) \right] \tag{14}$$

We will define the variable κ to be the continuous-time risk-adjusted discount rate. The equation for the discount rate is...

$$\kappa = \ln\left(1 + \text{annualized risk-adjusted discount rate}\right) \tag{15}$$

We will define the variable $PVIBL$ to be the present value (i.e. market value) of interest-bearing liabilities at time zero. Using Equations (11), (13) and (15) above the equation for the market value of interest-bearing liabilities at time zero is...

$$PVIBL = \int_0^T C_t \operatorname{Exp}\left\{-\kappa \, t\right\} \delta t + C_T \operatorname{Exp}\left\{-\kappa \, T\right\} \tag{16}$$

We will define the variable $PVIBL(0,T)$ to be the present value at time zero of lender cash flow prior to maturity and the variable $PVIBL(T)$ to be the present value at time zero of lender cash flow at maturity. Using these definitions we can rewrite Equation (16) above as...

$$PVIBL = PVIBL(0,T) + PVIBL(T) \tag{17}$$

Using Equations (12), (16) and (17) above the equation for the present value at time zero of lender cash flow prior to maturity is...

$$PVIBL(0,T) = \int_0^T (\alpha - \mu) \, \phi \, \theta R_0 \operatorname{Exp}\left\{\mu \, t\right\} \operatorname{Exp}\left\{-\kappa \, t\right\} \delta t = (\alpha - \mu) \, \phi \, \theta R_0 \int_0^T \operatorname{Exp}\left\{(\mu - \kappa) \, t\right\} \delta u \tag{18}$$

Using Appendix Equation (39) below the solution to Equation (18) above is...

$$PVIBL(0,T) = \frac{\alpha - \mu}{\mu - \kappa} \, \phi \, \theta R_0 \operatorname{Exp}\left\{(\mu - \kappa) \, u\right\} \Big|_0^T = \frac{\alpha - \mu}{\kappa - \mu} \, \phi \, \theta R_0 \left(1 - \operatorname{Exp}\left\{(\mu - \kappa) \, T\right\}\right) \tag{19}$$

Note that the equation for the limit of Equation (19) above as the liability term goes to infinity is...

$$PVIBL(0,\infty) = \lim_{T \to \infty} PVIBL(0,T) = \frac{\alpha - \mu}{\kappa - \mu} \, \phi \, \theta R_0 \quad \text{...when...} \quad \kappa > \mu \tag{20}$$

Using Equations (14), (16) and (17) above the equation for the present value at time zero of the principal and interest payment at maturity is...

$$PVIBL(T) = \phi \, \theta R_0 \left[\operatorname{Exp}\left\{\mu \, T\right\} + \frac{\omega - \alpha}{\mu - \omega} \left(\operatorname{Exp}\left\{\mu \, T\right\} - \operatorname{Exp}\left\{\omega \, T\right\} \right) \right] \operatorname{Exp}\left\{-\kappa \, T\right\} \tag{21}$$

Note that the equation for the limit of Equation (21) above as the term goes to infinity is...

$$PVIBL(\infty) = \lim_{T \to \infty} PVIBL(T) = 0 \quad \text{...when...} \quad \kappa > \mu \quad \text{...because...} \quad \lim_{T \to \infty} \operatorname{Exp}\left\{-\kappa \, T\right\} = 0 \tag{22}$$

Using Equations (19) and (21) above we can rewrite Equation (17) above is...

$$PVIBL = \phi \, \theta R_0 \left[\frac{\alpha - \mu}{\kappa - \mu} \left(1 - \operatorname{Exp}\left\{(\mu - \kappa) \, T\right\}\right) + \left[\operatorname{Exp}\left\{\mu \, T\right\} + \frac{\omega - \alpha}{\mu - \omega} \left(\operatorname{Exp}\left\{\mu \, T\right\} - \operatorname{Exp}\left\{\omega \, T\right\} \right) \right] \operatorname{Exp}\left\{-\kappa \, T\right\} \right] \tag{23}$$

Note that using Equations (20) and (22) above the equation for the limit of Equation (23) above as the term goes to infinity is...

$$\lim_{T \to \infty} PVIBL = \frac{\alpha - \mu}{\kappa - \mu} \, \phi \, \theta R_0 \tag{24}$$

NIBL Valuation Equations

When the revenue growth rate is positive then NIBL are an addition to company value for the following reasons:

(1) NIBL are a source of cash for the company and are financed at a zero percent interest rate.

(2) If the company's revenue base is growing its NIBL are also growing such that the increase in NIBL technically never has to be paid back.

In the case of non-interest bearing liabilities the coupon rate is zero and the term is infinite. We will define the variable $PVNIBL$ to be the present value at time zero of non-interest bearing liabiities. Using Equation (24) above the equation for the value of non-interest bearing liabilities is...

$$PVNIBL = -\frac{\mu}{\kappa - \mu}\,\phi\,\theta R_0 \text{ ...given that... } \alpha = \text{coupon rate} = 0 \tag{25}$$

Insight - Per Equation (25) above if the revenue growth rate is zero then the value of non-interest bearing debt is also zero. The higher the revenue growth rate, the more negative is the value of non-interest bearing debt. The primary driver of the value of non-interest bearing debt is fact that the higher the revenue growth rate the higher are the funds that the borrower receives that never have to be paid back.

Note that the discrete-time version of continuous-time Equation (25) above is...

$$PVNIBL = -\frac{g\,\phi\,\theta R_0\,(1+g)}{k-g} \text{ ...when... } g = \text{annualized revenue growth rate ...and... } k = \text{annualized discount rate} \tag{26}$$

The Answers To Our Hypothetical Problem

Using Equation (1) above the equation for the continuous-time revenue growth rate is...

$$\mu = \ln\left(1 + 0.03\right) = 0.02956 \tag{27}$$

Using Equation (5) above the equation for the continuous-time debt interest accrual rate is...

$$\omega = \ln\left(1 + 0.04\right) = 0.03922 \tag{28}$$

Using Equation (6) above the equation for the continuous-time debt coupon payment rate is...

$$\alpha = \ln\left(1 + 0.02\right) = 0.01980 \tag{29}$$

Using Equation (15) above the equation for the continuous-time debt risk-adjusted discount rate is...

$$\kappa = \ln\left(1 + 0.06\right) = 0.05827 \tag{30}$$

Question 1: What is the balance of total liabilities at time zero?

Using Equation (4) above and the model assumptions above the answer to the question is...

$$\text{IBL/NIBL balance at time zero} = 0.45 \times 1.20 \times 1,000,000 = 540,000 \tag{31}$$

Question 2: What is the market value of IBL at time zero given that term is 15 years?

Using Equation (9) above and the model assumptions above the answer to the question is...

$$\begin{aligned} PVIBL = 0.45 \times 1.20 \times 1,000,000 \times &\left[\frac{0.01980 - 0.02956}{0.05827 - 0.02956} \times \left(1 - \text{Exp}\left\{(0.02956 - 0.05827) \times 15\right\}\right)\right. \\ &+ \left[\text{Exp}\left\{0.02956 \times 15\right\} + \frac{0.03922 - 0.01980}{0.02956 - 0.03922} \times \left(\text{Exp}\left\{0.02956 \times 15\right\} - \text{Exp}\left\{0.03922 \times 15\right\}\right)\right] \\ &\left. \times \text{Exp}\left\{-0.05827 \times 15\right\}\right] = 396,867 \end{aligned} \tag{32}$$

The discount on debt is...

$$\text{Discount} = 1 - \frac{396,867}{540,000} = 27\% \tag{33}$$

There is a discount on debt because the rate of return required by the market (6.00%) is greater than the contractual interest accrual rate (4.00%). The borrower is allowed to borrow at a below market rate for 15 years.

Question 3: What is the market value of IBL at time zero given that term is perpetual?

Using Equation (24) above and the model assumptions above the answer to the question is...

$$PVIBL = \frac{0.01980 - 0.02956}{0.05827 - 0.02956} \times 0.45 \times 1.20 \times 1,000,000 = -183,501 \tag{34}$$

The value of debt is negative because the coupon rate (2%) is less than the market rate of return (6%).

Question 4: What is the market value of NIBL at time zero given that term is perpetual?

Using Equation (25) above and the model assumptions above the answer to the question is...

$$PVNIBL = -\frac{0.02956}{0.05827 - 0.02956} \times 0.45 \times 1.20 \times 1,000,000 = -555,963 \tag{35}$$

The value of debt is negative (and more negative than the answer to Question 3 above) because the coupon rate is zero vs 2% in Question 3.

Appendix Equations

A. The solution to the following integral is...

$$\int_s^t \text{Exp}\left\{\lambda\,u\right\} \delta u = \lambda^{-1} \text{Exp}\left\{\lambda\,u\right\}\Big[_{u=s}^{u=t} = \lambda^{-1}\left(\text{Exp}\left\{\lambda\,t\right\} - \text{Exp}\left\{\lambda\,s\right\}\right) \tag{36}$$

B. The solution to the following integral is...

$$
\begin{aligned}
I_t &= \int_0^t (\omega - \alpha)\,\phi\,\theta R_0 \,\text{Exp}\left\{\mu\,u\right\} \text{Exp}\left\{\omega\,(t-u)\right\} \delta u \\
&= (\omega - \alpha)\,\phi\,\theta R_0 \int_0^t \text{Exp}\left\{\mu\,u\right\} \text{Exp}\left\{\omega\,t\right\} \text{Exp}\left\{-\omega\,u\right\} \delta u \\
&= (\omega - \alpha)\,\phi\,\theta R_0 \,\text{Exp}\left\{\omega\,t\right\} \int_0^t \text{Exp}\left\{(\mu - \omega)\,u\right\} \delta u
\end{aligned}
\tag{37}
$$

Using Equation (36) above the solution to Equation (37) above is...

$$
\begin{aligned}
I_t &= (\omega - \alpha)\,\phi\,\theta R_0 \,\text{Exp}\left\{\omega\,t\right\} (\mu - \omega)^{-1}\left(\text{Exp}\left\{(\mu-\omega)\,t\right\} - 1\right) \\
&= (\omega - \alpha)\left(\frac{\phi\,\theta R_0}{\mu - \omega}\right)\left(\text{Exp}\left\{\mu\,t\right\} - \text{Exp}\left\{\omega\,t\right\}\right)
\end{aligned}
\tag{38}
$$

C. The solution to the following integral is...

$$\text{Integral} = \int_s^t \text{Exp}\left\{x\,u\right\} \delta u = \frac{1}{x} \text{Exp}\left\{x\,u\right\}\Big[_{u=s}^{u=t} = \frac{1}{x}\left(\text{Exp}\left\{x\,t\right\} - \text{Exp}\left\{x\,s\right\}\right) \tag{39}$$